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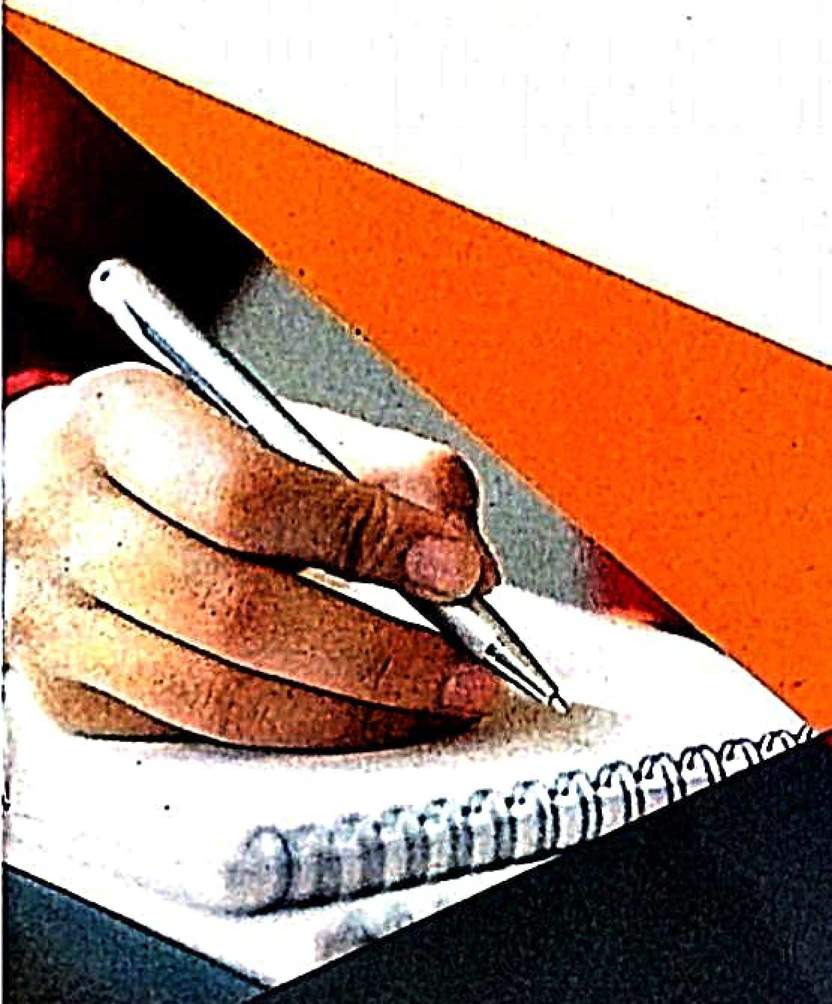
GROUP OF COLLEGE

Behind Kalwar Police Station, Kalwar, Jaipur (Raj.)



ASSIGNMENT

7.4 Morera's Theorem

Let f be a continuous function on an open set D. If

$$\int_{\Gamma} f(z)dz = 0$$

whenever Γ is the boundary of a closed rectangle in D, then f is analytic on D.

Since line integrals are unaffected by the value of the integrand at a single point, the continuity of f is a necessary hypothesis. Note also that in the proof, we actually require only that $\int_{\Gamma} f = 0$ for rectangles whose sides are parallel to the horizontal and vertical axes.

Proof

In a small disc about any point $z_0 \in D$, we can define a primitive

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$$

where the path of integration is the horizontal followed by the vertical segments from z_0 to z. If we then consider a difference quotient of F and apply the fact that $\int_{\Gamma} f = 0$ around any rectangle, we may conclude (as in Theorems 4.15 and 6.2) that

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{z}^{z+h} f(\zeta)d\zeta \to f(z)$$

as $h \to 0$. (Here we are using the continuity of f.) Hence F is analytic in a neighborhood of z_0. Since analytic functions are infinitely differentiable and $F'(z) = f(z)$, f is analytic at z_0. Finally, since z_0 was arbitrary, f is analytic in D. □

Morera's Theorem is often used to establish the analyticity of functions given in integral form. For example, consider

$$f(z) = \int_{0}^{\infty} \frac{e^{zt}}{t+1}\,dt.$$

If Re $z = x < 0$,

$$\int_{0}^{\infty} \frac{|e^{zt}|}{t+1}\,dt < \int_{0}^{\infty} e^{xt}\,dt = -\frac{1}{x}$$

so that the integral is absolutely convergent and $|f(z)| \leq 1/|x|$. To show that f is analytic in the left half-plane D : Re $z < 0$, we may consider

$$\int_{\Gamma} f(z)dz = \int_{\Gamma}\left(\int_{0}^{\infty} \frac{e^{zt}}{t+1}dt\right)dz,$$

where Γ is the boundary of some closed rectangle in D.

Since

$$\int_{\Gamma}\int_{0}^{\infty} \frac{|e^{zt}|}{t+1}dt\,dz$$

converges, we can interchange the order of integration; hence

$$\int_{\Gamma} f = \int_{0}^{\infty}\int_{\Gamma} \frac{e^{zt}}{t+1}dz\,dt = \int_{0}^{\infty} 0\,dt = 0$$

by the analyticity of $e^{zt}/(t+1)$ as a function of z. By Morera's Theorem, then, f is analytic in D.

6.13 Maximum-Modulus Theorem

A non-constant analytic function in a region D does not have any interior maximum points: For each $z \in D$ and $\delta > 0$, there exists some $\omega \in D(z;\delta) \cap D$, such that $|f(\omega)| > |f(z)|$.

Proof

The fact that

$$|f(\omega)| \geq |f(z)|$$

for some ω near z follows immediately from the Mean-Value Theorem. Since for $r > 0$ such that $D(z;r) \subset D$ we have

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})d\theta,$$

it follows that

$$|f(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z + re^{i\theta})|d\theta \leq \max_\theta |f(z + re^{i\theta})|. \tag{3}$$

Similarly, we may deduce that $|f(\omega)| > |f(z)|$ for some $\omega \in D(z;r)$. For, to obtain equality in (3), $|f|$ would have to be constant throughout the circle $C(z;r)$ and since this holds for all sufficiently small $r > 0$, $|f|$ would be constant throughout a disc. But then by Theorem 3.7, f would be constant in that disc, and by the Uniqueness Theorem, f would be constant throughout D. □

Ironically, the Maximum-Modulus Theorem actually asserts that an analytic function has no relative maximum. It is sometimes given a more positive flavor as follows.

Suppose a function f is analytic in a bounded region D and continuous on $\bar{D}$. (We will, henceforth, use the expression "f is C-analytic in D" to denote this hypothesis.) Somewhere in the compact domain $\bar{D}$, the continuous function $|f|$ must assume its maximum value. The Maximum-Modulus Theorem may then be invoked to assert that this maximum is always assumed on the boundary of the domain.

6.14 Minimum Modulus Theorem

If f is a non-constant analytic function in a region D, then no point $z \in D$ can be a relative minimum of f unless $f(z) = 0$.

Proof

Suppose that $f(z) \neq 0$ and consider $g = 1/f$. If z were a minimum point for f, it would be a maximum point for g. Hence g would be constant in D, contrary to our hypothesis on f. □

Remark

We can also prove the Maximum-Modulus Theorem by analyzing the local power series representation for an analytic function. That is, for any point α, consider the power series

$$f(z) = C_0 + C_1(z-\alpha) + C_2(z-\alpha)^2 + \cdots,$$

which is convergent in some disc around α. To find z near α and such that $|f(z)| > |f(\alpha)|$, we first assume $C_1 \neq 0$ and set $z = \alpha + \delta e^{i\theta}$, with $\delta > 0$ "small", and θ chosen so that C_0 and $C_1\delta e^{i\theta}$ have the same argument. Then

$$|f(\alpha)| = |C_0|$$

$$|f(z)| \geq |C_0 + C_1(z-\alpha)| - |C_2(z-\alpha)^2 + C_3(z-\alpha)^3 + \cdots|$$

$$\geq |C_0| + |C_1\delta| - \delta^2|C_2 + C_3(z-\alpha) + \cdots|.$$

Since the last expression represents a convergent series,

$$|f(z)| \geq |C_0| + |C_1\delta| - A\delta^2 \geq |C_0| + \frac{1}{2}|C_1\delta| > |f(\alpha)|$$

as long as $\delta < |C_1|/2A$. Hence α cannot be a maximum point. Note that if $C_1 = 0$, the same argument can be applied by focusing on the first non-zero coefficient C_k.

This technique of studying the local behavior of an analytic function by considering the first terms of its power series expansion can be used to derive the following result.

Recall that in calculus, relative maximum points were found among the critical points (those points at which $f' = 0$) of a differentiable function f. The proposition below shows a somewhat surprising contrast in the behavior of an analytic function at a point where it assumes its maximum modulus.

6.5 Power Series Representation for Functions Analytic in a Disc

If f is analytic in $D(\alpha; r)$ there exist constants C_k such that

$$f(z) = \sum_{k=0}^{\infty} C_k (z - \alpha)^k$$

for all $z \in D(\alpha; r)$.

Proof

Pick $a \in D(\alpha; r)$ and $\rho > 0$ such that $|a - \alpha| < \rho < r$.

By the previous integral formula, if $|z - \alpha| < |a - \alpha|$

$$f(z) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(\omega)}{\omega - z}\, d\omega$$

and using the fact that

$$\frac{1}{\omega - \alpha} + \frac{z - \alpha}{(\omega - \alpha)^2} + \frac{(z - \alpha)^2}{(\omega - \alpha)^3} + \cdots$$

converges uniformly to $1/(\omega - z)$ throughout C_ρ (see Lemma 5.4)

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_\rho} f(\omega) \left[\frac{1}{\omega - \alpha} + \frac{z - \alpha}{(\omega - \alpha)^2} + \frac{(z - \alpha)^2}{(\omega - \alpha)^3} + \cdots \right] d\omega \\ &= C_0(\rho) + C_1(\rho)(z - \alpha) + C_2(\rho)(z - \alpha)^2 + \cdots \qquad (1) \end{aligned}$$

where

$$C_k(\rho) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(\omega)}{(\omega - \alpha)^{k+1}}\, d\omega.$$

Note, then, that the coefficients $C_k(\rho)$ are actually independent of ρ. For once again, as in 5.5, we can apply (1) to conclude that f is infinitely differentiable at α and

$$C_k(\rho) = \frac{f^{(k)}(\alpha)}{k!} \text{ for each } \rho, 0 < \rho < r, \text{ and all } k.$$

Hence, for *all* $z \in D(\alpha; r)$

$$f(z) = \sum_{k=0}^{\infty} C_k (z - \alpha)^k$$

with

$$C_k = \frac{f^{(k)}(\alpha)}{k!} = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{(z - \alpha)^{k+1}}\, dz. \qquad \square$$

2.12 Uniqueness Theorem for Power Series

Suppose $\sum_{n=0}^{\infty} C_n z^n$ *is zero at all points of a nonzero sequence* $\{z_k\}$ *which converges to zero. Then the power series is identically zero.*

[Note: If we set $f(z) = \sum C_n z^n$, it follows from the continuity of power series that $f(0) = 0$. We can show by a similar argument that $f'(0) = 0$; however, a slightly different argument is needed to show that the higher coefficients are also 0.]

Proof

Let

$$f(z) = C_0 + C_1 z + C_2 z^2 + \cdots .$$

By the continuity of f at the origin

$$C_0 = f(0) = \lim_{z\to 0} f(z) = \lim_{k\to\infty} f(z_k) = 0.$$

But then

$$g(z) = \frac{f(z)}{z} = C_1 + C_2 z + C_3 z^2 + \cdots$$

is also continuous at the origin and

$$C_1 = g(0) = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{k\to\infty} \frac{f(z_k)}{z_k} = 0.$$

Similarly, if $C_j = 0$ for $0 \le j < n$, then

$$C_n = \lim_{z\to 0} \frac{f(z)}{z^n} = \lim_{k\to\infty} \frac{f(z_k)}{z_k^n} = 0,$$

so that the power series is identically zero. □